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Ionization and Recombination in Plasmas

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Ionization and recombination in a plasma are considered. They give rise to a system of kinetic equations with cubic source terms. An existence theorem is proved for such a system.

KEY WORDS: Plasmas; ionization; recombination; kinetic system of equations; cubic collision operator.

INTRODUCTION

Discharges on satellites are a recent domain of interest. They have been observed at low altitude, and give rise to electron beams[5, 12] which degrade the satellite's efficiency. The discharges propagate in a plasma created by the ionization of desorbed neutral molecules.[10] The processes of ionization and recombination have already been described, among other collisions, in ref. 9, and numerically simulated in refs. 13 and 14. In this paper, we focus on the ionization and recombination processes arising in such sparse plasmas. We will not take into account the excitations of molecules going along with such phenomena.[7] Ionization of a molecule colliding with an electron and giving rise to an ion and two electrons is considered, as well as the reciprocal recombination phenomenon. Kinetic equations are more appropriate than fluid ones because of the sparsity of the considered plasmas. The source terms contain cubic terms. Cubic terms also arise in the study of nondegenerate semiconductors.[11] By Fermi's principle, however, the distribution function there is known to be bounded. Multiple collisions have been taken into account in the Boltzmann context with discrete and semidiscrete models.[2, 4] Here, analogously to the Boltzmann equation, the presence of cubic terms

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gives rise to the mathematical problem of giving them a sense, since the distribution functions are only expected to belong to $L^1 \log L^1$.[1,6] In this paper, we overcome this difficulty by assuming that a molecule and a subsequent ion in an ionizing collision have the same velocity.

1. THE IONIZATION–RECOMBINATION SYSTEM AND *A PRIORI* ESTIMATES

Let $\bar{\alpha}(v_1, v_2, v_3, v_4, v_5)$ be the transition rate from an ion and two electrons with respective velocities v_1, v_2, and v_3 to an electron and a molecule with respective velocities v_4 and v_5. Then,

$$\bar{\alpha}(v_1, v_2, v_3, v_4, v_5) = \bar{\alpha}(v_1, v_3, v_2, v_4, v_5), \qquad (v_i) \in \mathbb{R}^{15}, \quad 1 \leqslant i \leqslant 5 \tag{1.1}$$

By the conservations of momentum and energy, the support of $\bar{\alpha}$ is the set

$$M_i v_1 + M_e(v_2 + v_3) = M_e v_4 + M_m v_5 \tag{1.2}$$

$$\tfrac{1}{2} M_i v_1^2 + \tfrac{1}{2} M_e(v_2^2 + v_3^2) = \tfrac{1}{2} M_e v_4^2 + \tfrac{1}{2} M_m v_5^2 + \bar{\Delta} \tag{1.3}$$

where M_i, M_e, M_m, and $\bar{\Delta}$, respectively, denote the ion, electron, and molecule masses and the ionization energy. Let $f(t, x, v)$, $g(t, x, v)$, and $h(t, x, v)$, respectively, denote the ion, electron, and molecule distribution functions. We only consider collisions between particles of different types. Hence the ionization–recombination system is

$$\begin{aligned} f_t + v \cdot \nabla_x f &= -f(t, x, v) \int \bar{\alpha}(v, v_2, v_3, v_4, v_5)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{2345} \\ &\quad + \int \bar{\alpha}(v, v_2, v_3, v_4, v_5)\, g(t, x, v_4)\, h(t, x, v_5)\, dv_{2345} \end{aligned} \tag{1.4}$$

$$\begin{aligned} g_t + v \cdot \nabla_x g &= -g(t, x, v) \left[2 \int \bar{\alpha}(v_1, v, v_3, v_4, v_5)\, f(t, x, v_1)\, g(t, x, v_3)\, dv_{1345} \right. \\ &\quad \left. + \int \bar{\alpha}(v_1, v_2, v_3, v, v_5)\, h(t, x, v_5)\, dv_{1235} \right] \\ &\quad + \int \bar{\alpha}(v_1, v_2, v_3, v, v_5)\, f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{1235} \\ &\quad + 2 \int \bar{\alpha}(v_1, v, v_3, v_4, v_5)\, g(t, x, v_4)\, h(t, x, v_5)\, dv_{1345} \end{aligned} \tag{1.5}$$

$$h_t + v \cdot \nabla_x h$$
$$= -h(t, x, v) \int \bar{\alpha}(v_1, v_2, v_3, v_4, v)\, g(t, x, v_4)\, dv_{1234}$$
$$+ \int \bar{\alpha}(v_1, v_2, v_3, v_4, v)\, f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{1234} \quad (1.6)$$

This system will be considered in a periodic box Λ, with nonnegative initial data $(f_0, g_0, h_0) = (f(0), g(0), h(0))$ such that

$$f_0(1 + |v|^2 + \ln f_0) \in L^1_+, \qquad g_0(1 + |v|^2 + \ln g_0) \in L^1_+$$
$$h_0(1 + |v|^2 + \ln h_0) \in L^1_+ \quad (1.7)$$

Let us first notice that nonnegativity is preserved, i.e., f, g, and h are nonnegative like f_0, g_0, and h_0. Then, adding (1.4) and (1.6) and integrating over $(0, t) \times \Lambda \times \mathbb{R}^3_v$ gives

$$\int_{\Lambda \times \mathbb{R}^3_v} (f + h)(t, x, v)\, dx\, dv = \int_{\Lambda \times \mathbb{R}^3_v} (f_0 + h_0)(x, v)\, dx\, dv$$

so that, by (1.7),

$$\int_{\Lambda \times \mathbb{R}^3_v} (f + h)(t, x, v)\, dx\, dv < \infty, \qquad t \in \mathbb{R}^+ \quad (1.8)$$

Analogously, adding (1.5) and (1.6) and using (1.7) gives

$$\int_{\Lambda \times \mathbb{R}^3_v} (g + h)(t, x, v)\, dx\, dv < \infty, \qquad t \in \mathbb{R}^+ \quad (1.9)$$

Moreover, adding (1.4)–(1.6) and integrating over $(0, t) \times \Lambda \times \mathbb{R}^3_v$ yields

$$\int_{\Lambda \times \mathbb{R}^3_v} (f + g + h)(t, x, v)\, dx\, dv$$
$$= \int_{\Lambda \times \mathbb{R}^3_v} (f_0 + g_0 + h_0)(x, v)\, dx\, dv - \int_{(0, t) \times \Lambda \times \mathbb{R}^{15}_{v_i}} \bar{\alpha}(v_1, v_2, v_3, v_4, v_5)$$
$$\times (f(s, x, v_1)\, g(s, x, v_2)\, g(s, x, v_3) - g(s, x, v_4)\, h(s, x, v_5))\, dv_i\, dx\, ds \quad (1.10)$$

Then, multiplying (1.4)–(1.6) by $M_i\,|v|^2$, $M_e\,|v|^2$, and $M_m\,|v|^2$, respectively, and integrating over $(0, t)\times \Lambda \times \mathbb{R}^3_v$ gives

$$\int_{\Lambda\times\mathbb{R}^3_v} (M_i f + M_e g + M_m h)(t, x, v)\,|v|^2\,dx\,dv$$

$$= \int_{\Lambda\times\mathbb{R}^3_v} (M_i f_0 + M_e g_0 + M_m h_0)(x, v)\,|v|^2\,dx\,dv$$

$$+ \int_{(0,t)\times\Lambda\times\mathbb{R}^{15}_{v_i}} \bar{\alpha}(v_1, v_2, v_3, v_4, v_5)[f(s, x, v_1)\,g(s, x, v_2)\,g(s, x, v_3)$$

$$- g(s, x, v_4)\,h(s, x, v_5)]$$

$$\times(-M_i v_1^2 - M_e(v_2^2 + v_3^2 - v_4^2) + M_m v_5^2)\,dv_i\,dx\,ds, \qquad t\in\mathbb{R}^+ \qquad (1.11)$$

By (1.3), the right-hand side of (1.11) is

$$\int_{\Lambda\times\mathbb{R}^3} (M_i f_0 + M_e g_0 + M_m h_0)(x, v)\,|v|^2\,dx\,dv$$

$$-2\bar{\Delta}\int_{(0,t)\times\Lambda\times\mathbb{R}^{15}_{v_i}} \bar{\alpha}(v_1, v_2, v_3, v_4, v_5)$$

$$\times(f(s, x, v_1)\,g(s, x, v_2)\,g(s, x, v_3) - g(s, x, v_4)\,h(s, x, v_5))\,dv_i\,dx\,ds$$

which, by (1.10), is equal to

$$\int_{\Lambda\times\mathbb{R}^3} (M_i f_0 + M_e g_0 + M_m h_0)(x, v)\,|v|^2\,dx\,dv$$

$$+ 2\bar{\Delta}\int_{\Lambda\times\mathbb{R}^3} (f + g + h)(t, x, v)\,dx\,dv$$

$$- 2\bar{\Delta}\int_{\Lambda\times\mathbb{R}^3} (f_0 + g_0 + h_0)(x, v)\,dx\,dv$$

Hence

$$\int_{\Lambda\times\mathbb{R}^3} (f + g + h)(t, x, v)\,|v|^2\,dx\,dv < \infty, \qquad t\in\mathbb{R}^+ \qquad (1.12)$$

Finally, multiplying (1.4) by $\ln f$, (1.5) by $\ln g$, and (1.6) by $\ln h$ and integrating over $(0, t) \times \Lambda \times \mathbb{R}^3$ gives

$$\int_{\Lambda \times \mathbb{R}^3} (f \ln f + g \ln g + h \ln h)(t, x, v)\, dx\, dv$$
$$+ \int_{(0,t) \times \Lambda \times \mathbb{R}^{15}_{v_i}} \bar{\alpha}(v_1, v_2, v_3, v_4, v_5)[f(s, x, v_1)\, g(s, x, v_2)\, g(s, x, v_3)$$
$$- g(s, x, v_4)\, h(s, x, v_5)]$$
$$\times \ln \frac{f(s, x, v_1)\, g(s, x, v_2)\, g(s, x, v_3)}{g(s, x, v_4)\, h(s, x, v_5)}\, dv_i\, dx\, ds$$
$$= \int_{\Lambda \times \mathbb{R}^3} (f_0 \ln f_0 + g_0 \ln g_0 + h_0 \ln h_0)(x, v)\, dx\, dv \tag{1.13}$$

It follows from (1.7), the nonnegativity of the second term of the left-hand side of (1.13), and a classical argument[3] that

$$\int_{\Lambda \times \mathbb{R}^3} (f\, |\ln f| + g\, |\ln g| + h\, |\ln h|)(t, x, v)\, dx\, dv < \infty, \qquad t \in \mathbb{R}^+ \tag{1.14}$$

and

$$\int_{\mathbb{R}^+ \times \Lambda \times \mathbb{R}^{15}_{v_i}} \bar{\alpha}(v_1, v_2, v_3, v_4, v_5)[f(s, x, v_1)\, g(s, x, v_2)\, g(s, x, v_3)$$
$$- g(s, x, v_4)\, h(s, x, v_5)]$$
$$\times \ln \frac{f(s, x, v_1)\, g(s, x, v_2)\, g(s, x, v_3)}{g(s, x, v_4)\, h(s, x, v_5)}\, dv_i\, dx\, ds < \infty \tag{1.15}$$

Remark. The *a priori* estimates (1.8), (1.9), (1.12), (1.14), and (1.15) also hold when Λ is a bounded set, for given ingoing data as boundary conditions.

2. EQUAL ION AND MOLECULE VELOCITIES. DEFINITION OF A SOLUTION

Taking into account that $M_e \ll M_i$, we can approximate the momentum and energy conservation laws by

$$v_1 = v_5$$
$$M_e(v_2^2 + v_3^2) = M_e v_4^2 + 2\bar{\Delta}$$

Hence we consider transition rates $\bar{\alpha}$ of the following type:

(A1) $\bar{\alpha}(v_1, v_2, v_3, v_4, v_5) = \alpha(v_1, v_2, v_3, v_4, v_5)\, \delta_S$

where S denotes

$$S = \{(v_i) \in \mathbb{R}^{15}/v_1 = v_5,\ M_e(v_2^2 + v_3^2) = M_e v_4^2 + 2\bar{\Delta}\}$$

Denote $\Delta := 2\bar{\Delta}/M_e$, and

$$Z(v_2, v_3, v_4) := \{(v_2, v_3, v_4)/v_2^2 + v_3^2 = v_4^2 + \Delta\}$$

We assume that α is a nonnegative measurable function satisfying

(A2) $\int_{Z(v_2, v_3, v_4)} \alpha(v_1, v_2, v_3, v_4, v_1)\, dv_2\, dv_3 \in L^\infty(\mathbb{R}^6_{v_{14}})$

(A3) $\int_{Z(v_2, v_3, v_4)} \alpha(v_1, v_2, v_3, v_4, v_1)\, dv_4 \in L^\infty_{v_3}(L^1_{v_{12}}) \cap L^\infty_{v_1}(L^1_{v_{23}})$

Moreover, assume that the initial data are nonnegative and satisfy (1.7) and

(A8) $f_0 \in L^1(\mathbb{R}^3_v; L^\infty(\Lambda_x))$, $g_0 \in L^1(\Lambda_x \times \mathbb{R}^3_v)$, $h_0 \in L^1(\mathbb{R}^3_v; L^\infty(\Lambda_x))$

(A9) $\forall \varepsilon > 0,\ \exists \eta > 0$: $C \subset \mathbb{R}^3$ and $|C| < \eta \Rightarrow \|f_0 + h_0\|_{L^1(C; L^\infty(\Lambda))} < \varepsilon$

It follows from (A1) that the ionization–recombination system (1.4)–(1.6) is

$$\begin{aligned} f_t &+ v \cdot \nabla_x f \\ &= -f(t, x, v) \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{234} \\ &\quad + h(t, x, v) \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_4)\, dv_{234} \end{aligned} \tag{2.1}$$

$$\begin{aligned} g_t &+ v \cdot \nabla_x g \\ &= -g(t, x, v) \Bigg[2 \int_{\mathbb{R}^3_{v_1} \times Z(v, v_3, v_4)} \alpha(v_1, v, v_3, v_4, v_1)\, f(t, x, v_1) \\ &\quad \times g(t, x, v_3)\, dv_{134} \\ &\quad + \int_{\mathbb{R}^3_{v_5} \times Z(v_2, v_3, v)} \alpha(v_5, v_2, v_3, v, v_5)\, h(t, x, v_5)\, dv_{235} \Bigg] \\ &\quad + \int_{\mathbb{R}^3_{v_1} \times Z(v_2, v_3, v)} \alpha(v_1, v_2, v_3, v, v_1)\, f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{123} \\ &\quad + 2 \int_{\mathbb{R}^3_{v_5} \times Z(v, v_3, v_4)} \alpha(v_5, v, v_3, v_4, v_5)\, g(t, x, v_4)\, h(t, x, v_5)\, dv_{345} \end{aligned} \tag{2.2}$$

$$h_t + v \cdot \nabla_x h$$

$$= f(t, x, v) \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{234}$$

$$- h(t, x, v) \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_4)\, dv_{234} \tag{2.3}$$

Adding (2.1) and (2.3) leads to

$$(f + h)_t + v \cdot \nabla_x (f + h) = 0$$

so that, by (A4),

$$\int (f + h)(t, x, v)\, dv = \int (f_0 + h_0)(x - tv, v)\, dv < \infty \tag{2.4}$$

Hence h belongs to $L^\infty(\mathbb{R}^+ \times \Lambda_x; L^1(\mathbb{R}^3_v))$. Moreover, by (A2) and (1.9),

$$\int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_4)\, dv_{234}$$

belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^3_v, L^1(\Lambda_x))$.

Hence their product is well defined. Then,

$$\int_{\mathbb{R}^+ \times \Lambda \times \mathbb{R}^3} f(t, x, v)$$

$$\times \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{234}\, dv\, dx\, dt$$

$$= \int_{\mathbb{R}^+ \times \Lambda \times \mathbb{R}^{15}} \bar{\alpha}(v_i)\, f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_i\, dx\, dt$$

$$\leqslant j \int_{\mathbb{R}^+ \times \Lambda \times \mathbb{R}^{15}} \bar{\alpha}(v_i)\, g(t, x, v_4)\, h(t, x, v_5)\, dv_i\, dx\, dt$$

$$+ \frac{1}{\ln j} \int_{\mathbb{R}^+ \times \Lambda \times \mathbb{R}^{15}} \bar{\alpha}(v_i)[f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)$$

$$- g(t, x, v_4)\, h(t, x, v_5)]$$

$$\times \ln \frac{f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)}{g(t, x, v_4)\, h(t, x, v_5)}\, dv_i\, dx\, dt < \infty, \qquad j > 1 \tag{2.5}$$

by the last argument and (1.15). Hence

$$f(t, x, v) \int_{Z(r_2, r_3, r_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_2)\, g(t, x, v_3)\, dv_{234}$$

and

$$h(t, x, v) \int_{Z(r_2, r_3, r_4)} \alpha(v, v_2, v_3, v_4, v)\, g(t, x, v_4)\, dv_{234}$$

belong to L^1. And so the usual solutions f and h of (2.1) and (2.3) in the distributional sense can be considered. Analogously, by (1.9), (2.4), and (A2)

$$g(t, x, v) \int_{\mathbb{R}^3_{r_5} \times Z(r_2, r_3, r)} \alpha(v_5, v_2, v_3, v, v_5)\, h(t, x, v_5)\, dv_{235}$$

and

$$\int_{\mathbb{R}^3_{r_5} \times Z(r, r_3, r_4)} \alpha(v_5, v, v_3, v_4, v_5)\, g(t, x, v_4)\, h(t, x, v_5)\, dv_{345}$$

belong to L^1. Similar arguments to (2.5) prove that the right-hand side of (2.2) belongs to L^1, so that the usual solution g of (2.2) in the distribution sense can be considered.

Remark. We have just proved that our equations for electrons, ions, and neutral molecules can be treated in the distributional sense, in contrast to the DiPerna–Lions case for a single-species gas. This is so because our collision terms are in L^1, whereas the DiPerna–Lions collision term is only in L^1 after renormalization.

3. APPROXIMATE SOLUTIONS

A truncated version of the ionization–recombination system is used. For any positive integer n, let

$$\langle f \rangle_n = f \wedge n \qquad [\equiv \min(f, n)]$$

$$\alpha_n(v_1, v_2, v_3, v_4, v_5) = \alpha(v_1, v_2, v_3, v_4, v_5)\, \chi_{r_4^2 + A \leqslant n}$$

$$f_0^n = f_0 \wedge n, \qquad g_0^n = g_0 \wedge n, \qquad h_0^n = h_0 \wedge n$$

Here, χ_X denotes the characteristic function of the set X. Let the truncated ionization–recombination system be

$$
\begin{aligned}
f_t &+ v \cdot \nabla_x f \\
&= -(h_t + v \cdot \nabla_x h) \\
&= -\int_{Z(r_2, r_3, r_4)} \alpha_n(v, v_2, v_3, v_4, v) \langle f(v)\, g(v_2)\, g(v_3) \rangle_n \, dv_{234} \\
&\quad + \int_{Z(r_2, r_3, r_4)} \alpha_n(v, v_2, v_3, v_4, v) \langle g(v_4)\, h(v) \rangle_n \, dv_{234} \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
g_t &+ v \cdot \nabla_x g \\
&= -2 \int_{\mathbb{R}^3_{r_1} \times Z(r, r_3, r_4)} \alpha_n(v_1, v, v_3, v_4, v_1) \langle f(v_1)\, g(v)\, g(v_3) \rangle_n \, dv_{134} \\
&\quad + \int_{\mathbb{R}^3_{r_5} \times Z(r_2, r_3, r)} \alpha_n(v_5, v_2, v_3, v, v_5) \langle g(v)\, h(v_5) \rangle_n \, dv_{235} \\
&\quad + \int_{\mathbb{R}^3_{r_1} \times Z(r_2, r_3, r)} \alpha_n(v_1, v_2, v_3, v, v_1) \langle f(v_1)\, g(v_2)\, g(v_3) \rangle_n \, dv_{123} \\
&\quad + 2 \int_{\mathbb{R}^3_{r_5} \times Z(r, r_3, r_4)} \alpha_n(v_5, v, v_3, v_4, v_5) \langle g(v_4)\, h(v_5) \rangle_n \, dv_{345} \qquad (3.2)
\end{aligned}
$$

with solutions f^n, g^n, and h^n taking f^n_0, g^n_0, and h^n_0 as initial values. The existence of f^n, g^n, and h^n is proved by a Banach fixed-point argument in $C(0, T; L^1(\Lambda \times \mathbb{R}^3))$ for any $T > 0$, hence in $C(\mathbb{R}_+; L^1(\Lambda \times \mathbb{R}^3))$. Moreover, f^n, g^n, and h^n belong to L^∞_+ (with an L^∞-bound depending on n). By similar arguments to Section 1, they satisfy

$$
\begin{aligned}
&\int_{\Lambda \times \mathbb{R}^3} (f^n + g^n + h^n)(t, x, v)(1 + |v|^2)\, dx\, dv < c \\
&\int_{\Lambda \times \mathbb{R}^3} f^n(t, x, v)(1 + |v|^2 + |\ln f^n(t, x, v)|)\, dx\, dv < c \\
&\int_{\Lambda \times \mathbb{R}^3} g^n(t, x, v)(1 + |v|^2 + |\ln g^n(t, x, v)|)\, dx\, dv < c \qquad (3.3) \\
&\int_{\Lambda \times \mathbb{R}^3} h^n(t, x, v)(1 + |v|^2 + |\ln h^n(t, x, v)|)\, dx\, dv < c, \qquad t \in \mathbb{R}_+
\end{aligned}
$$

and

$$0 \leqslant \int_0^{+\infty} \int_{\Lambda \times \mathbb{R}^{15}} \alpha_n(v_i)(\langle f^n(v_1)\, g^n(v_2)\, g^n(v_3)\rangle_n - \langle g^n(v_4)\, h^n(v_5)\rangle_n)$$

$$\times \ln \frac{f^n(v_1)\, g^n(v_2)\, g^n(v_3)}{g^n(v_4)\, h^n(v_5)}\, dv_i\, dx\, dt < c \tag{3.4}$$

Here c are constants independent of n. It follows from (3.3) that (f^n), (g^n), and (h^n) are weakly compact in $L^1(\Lambda \times \mathbb{R}^3)$ for $t \geqslant 0$ and in $L^1((0, T) \times \Lambda \times \mathbb{R}^3)$ for $T > 0$.

Proposition 3.1. Let f^n, g^n, and h^n be solutions to (3.1)–(3.2). Then, for any $\psi \in L^\infty((0, T) \times \Lambda \times \mathbb{R}^3; L^1(\mathbb{R}^3))$,

$$\int f^n(t, x, v)\, \psi(t, x, v, w)\, dv$$

$$\int g^n(t, x, v)\, \psi(t, x, v, w)\, dv$$

and

$$\int h^n(t, x, v)\, \psi(t, x, v, w)\, dv$$

belong to a compact set in $L^1((0, T) \times \Lambda \times \mathbb{R}^3_w)$.

The averaging technique of Golse *et al.*[8] will be used in the form stated in ref. 6.

Lemma 3.2. Let (E, μ) be an arbitrary measure space, and let $\psi \in L^\infty((0, T) \times \Lambda \times \mathbb{R}^3; L^1(E))$.

(i) If f^n and F^n belong to a weakly compact set in $L^1(K)$ for any compact set in $(0, T) \times \Lambda \times \mathbb{R}^3$, and $f^n_t + v \cdot \nabla_x f^n = F^n$ in the distribution sense, then $\int_{\mathbb{R}^3} f^n \psi\, dv$ belongs to a compact set in $L^1((0, T) \times \Lambda \times E)$ for any compact set K in $(0, T) \times \Lambda \times \mathbb{R}^3$, provided that supp $\psi \subset K \times E$.

(ii) If in addition f^n belongs to a weakly compact set in $L^1((0, T) \times \Lambda \times \mathbb{R}^3)$, then $\int_{\mathbb{R}^3} f^n \psi\, dv$ belongs to a compact set in $L^1((0, T) \times \Lambda \times E)$.

For a proof of Lemma 3.2, see refs. 6 and 8.

Proof of Proposition 3.1. Given the weak L^1 compactness of f^n, g^n, and h^n, it remains to prove that the right-hand sides of (3.1)–(3.3) are locally L^1 compact in the sense of Lemma 3.2. We have that

$$\int_{Z(v_2, v_3, v_4)} \alpha_n(v, v_2, v_3, v_4, v) \langle g^n(t, x, v_4)\, h^n(t, x, v) \rangle_n \, dv_{234}$$

is bounded in L^1_{loc}, since

$$\int_{A \times \mathbb{R}^3_v} \int_{Z(v_2, v_3, v_4)} \alpha_n(v, v_2, v_3, v_4, v) \langle g^n(t, x, v_4)\, h^n(t, x, v) \rangle_n \, dv_{234} \, dx \, dv$$

$$\leqslant \|h^n\|_{L^{\infty}_{t,x}(L^1_v)} \left\| \int_{Z(v_2, v_3, v_4)} \alpha(v, v_2, v_3, v_4, v) \, dv_{23} \right\|_{L^{\infty}_{v, v_4}} \|g^n\|_{L^{\infty}_t(L^1_{x,v})}$$

Moreover, for any measurable set $B \times C$, where $B \subset \mathbb{R}^+_t \times A$ and $C \subset \mathbb{R}^3_v$,

$$\int_{B \times C} \int_{Z(v_2, v_3, v_4)} \alpha_n(v, v_2, v_3, v_4, v) \langle g^n(t, x, v_4)\, h^n(t, x, v) \rangle_n \, dv_{234} \, dt \, dx \, dv$$

$$= \int_{B \times C} h^n(t, x, v) \left[\int_{Z(v_2, v_3, v_4)} \alpha_n(v, v_2, v_3, v_4, v) \times g^n(t, x, v_4) \, dv_{234} \right] dt \, dx \, dv$$

is arbitrarily small when $|B \times C|$ is small enough. Indeed, the $L^1(\mathbb{R}^+_t \times A)$-weak compactness of

$$\int_{Z(v_2, v_3, v_4)} \beta_n(v, v_2, v_3, v_4, v) \, g^n(t, x, v_4) \, dv_{234}$$

follows from the weak L^1-compactness of (g^n) and (A2). Hence, if $|B|$ is small enough and C is bounded,

$$\int_{B \times C} h^n(t, x, v) \left[\int_{Z(v_2, v_3, v_4)} \alpha_n(v, v_2, v_3, v_4, v) \, g^n(t, x, v_4) \, dv_{234} \right] dt \, dx \, dv$$

$$\leqslant c \, \|h^n\|_{L^{\infty}(\mathbb{R}^+_t \times A;\, L^1(\mathbb{R}^3_v))} \int_{B \times \mathbb{R}^3_{v_4}} g^n(t, x, v_4) \, dv_4 \, dx \, dt$$

is arbitrarily small. Whereas if B is bounded and $|C|$ small enough,

$$\begin{aligned}
&\int_{B\times C} h''(t,x,v)\int_{Z(v_2,v_3,v_4)} \alpha_n(v,v_2,v_3,v_4,v)\, g''(t,x,v_4)\, dv_{234}\, dt\, dx\, dv \\
&\quad\leqslant c\int_{B\times C}(f_0+h_0)(x-tv,v)\int_{\mathbb{R}^3_{v_4}} g''(t,x,v_4)\, dv_4\, dt\, dx\, dv \\
&\quad\leqslant c\,\|f_0+h_0\|_{L^1(C;\,L^\infty(A))}\,\|g''\|_{L^\infty(\mathbb{R}^+;\,L^1(A\times\mathbb{R}^3))}
\end{aligned}$$

is arbitrarily small, by (A5). Since every small set in $\mathbb{R}^+\times A\times\mathbb{R}^3$ is small in the v direction outside of a small set in $\mathbb{R}^+\times A$, it follows that

$$\begin{aligned}
&\int_{A_{t,x,v}}\int_{Z(v_2,v_3,v_4)} \alpha_n(v,v_2,v_3,v_4,v) \\
&\qquad\times\langle g''(t,x,v_4)\, h''(t,x,v)\rangle_n\, dv_{234}\, dt\, dx\, dv
\end{aligned}$$

is arbitrarily small when $|A_{t,x,v}|$ is small enough. Then

$$\begin{aligned}
&\int_{Z(v_2,v_3,v_4)} \alpha_n(v,v_2,v_3,v_4,v)\langle f''(v)\, g''(v_2)\, g''(v_3)\rangle\, dv_{234} \\
&\quad\leqslant j\int_{Z(v_2,v_3,v_4)} \alpha_n(v,v_2,v_3,v_4,v)\langle g''(v_4)\, h''(v)\rangle_n\, dv_{234} \\
&\qquad+\frac{1}{\ln j}\int_{Z(v_2,v_3,v_4)} \alpha_n(v,v_2,v_3,v_4,v)[\langle f''(v)\, g''(v_2)\, g''(v_3)\rangle_n \\
&\qquad-\langle g''(v_4)\, h''(v)\rangle_n]\ln\frac{\langle f''(v)\, g''(v_2)\, g''(v_3)\rangle_n}{\langle g''(v_4)\, h''(v)\rangle_n}\, dv_{234}, \qquad j>1 \qquad (3.5)
\end{aligned}$$

so that the weak L^1-compactness of the first term of the right-hand side and the arbitrary smallness of the second term of (3.5) imply the weak L^1-compactness of

$$\int_{Z(v_2,v_3,v_4)} \beta_n(v,v_2,v_3,v_4,v)\langle f''(t,x,v)\, g''(t,x,v_2)\, g''(t,x,v_3)\rangle_n\, dv_{234}$$

The weak L^1-compactness of the right-hand sides of (3.2)–(3.3) follow from similar arguments.

4. AN EXISTENCE THEOREM

Theorem 4.1. Let (α, f_0, g_0, h_0) satisfy (A1)–(A5). Then there are solutions f, g, and h to the ionization–recombination system (2.1)–(2.3) in distribution sense on $\mathbb{R}^+ \times \Lambda \times \mathbb{R}^3_v$. Moreover, f and h belong to $L^\infty(\mathbb{R}^+ \times \Lambda; L^1(\mathbb{R}^3_v))$ and

$$\int_{\Lambda \times \mathbb{R}^3} f(t, x, v)(1 + |v|^2 + |\ln f|\ (t, x, v))\, dx\, dv < \infty$$

$$\int_{\Lambda \times \mathbb{R}^3} g(t, x, v)(1 + |v|^2 + |\ln g|\ (t, x, v))\, dx\, dv < \infty \tag{4.1}$$

$$\int_{\Lambda \times \mathbb{R}^3} h(t, x, v)(1 + |v|^2 + |\ln h|\ (t, x, v))\, dx\, dv < \infty, \qquad t \in \mathbb{R}^+$$

Proof of Theorem 4.1. By a Cantor diagonalization argument, it is enough to prove Theorem 4.1 on an arbitrary interval of time $(0, T)$. Let us pass to the limit in the weak formulation satisfied by h^n, namely

$$\begin{aligned}
&\int_{\Lambda \times \mathbb{R}^3} h^n(t)\, \psi(t) - \int_{\Lambda \times \mathbb{R}^3} h_0 \psi(0) - \int_0^T \int_{\Lambda \times \mathbb{R}^3} h^n(\psi_t + v \cdot \nabla\psi)(t) \\
&\quad = -\int_{(0, T) \times \Lambda} \int_{\mathbb{R}^3_{v_1} \times Z(v_2, v_3, v_4)} \psi(t, x, v_1)\, \alpha_n(v_{12341}) \\
&\qquad \times \langle g^n(t, x, v_4)\, h^n(t, x, v_1) \rangle_n\, dv_{1234}\, dt\, dx \\
&\qquad + \int_{(0, T) \times \Lambda} \int_{\mathbb{R}^3_{v_1} \times Z(v_2, v_3, v_4)} \psi(t, x, v_1)\, \alpha_n(v_{12341}) \\
&\qquad \times \langle f^n(t, x, v_1)\, g^n(t, x, v_2)\, g^n(t, x, v_3) \rangle_n\, dv_{1234}\, dt\, dx
\end{aligned} \tag{4.2}$$

where ψ is a compactly supported function belonging to $C^1((0, T) \times \Lambda; L^\infty(\mathbb{R}^3))$. It follows from the $L^1(\Lambda \times \mathbb{R}^3)$ weak compactness of $h^n(t, \cdot, \cdot)$, as well as the $L^1((0, T) \times \Lambda \times \mathbb{R}^3)$ weak compactness of h^n, that the left-hand side of (4.2) converges to

$$\int_{\Lambda \times \mathbb{R}^3} h(t)\, \psi(t) - \int_{\Lambda \times \mathbb{R}^3} h_0 \psi(0) - \int_0^T \int_{\Lambda \times \mathbb{R}^3} h(\psi_t + v \cdot \nabla\psi)(t)$$

when n tends to infinity. Moreover, the energy bound from (3.3) and the weak L^1-compactness of (g^n) imply that

$$\int_{(0,T)\times\Lambda}\int_{\mathbb{R}^3_{v_1}\times Z(v_2,v_3,v_4)} \psi(t,x,v_1)[\alpha_n(v_{12341})\langle g^n(t,x,v_4)\,h^n(t,x,v_5)\rangle_n - \alpha(v_{12341})\,g^n(t,x,v_4)\,h^n(t,x,v_5)]\,dv_{1234}\,dt\,dx$$

tends to zero when n tends to infinity. By (3.4) and similar arguments,

$$\int_{(0,T)\times\Lambda}\int_{\mathbb{R}^3_{v_1}\times Z(v_2,v_3,v_4)} \psi(t,x,v_1) \times[\alpha_n(v_{12341})\langle f^n(t,x,v_1)\,g^n(t,x,v_2)\,g^n(t,x,v_3)\rangle_n - \alpha(v_{12341})\,f^n(t,x,v_1)\,g^n(t,x,v_2)\,g^n(t,x,v_3)]\,dv_{1234}\,dt\,dx$$

tends to zero when n tends to infinity. Therefore it remains to pass to the limit in

$$\int_{(0,T)\times\Lambda}\int_{\mathbb{R}^3_{v_1}\times Z(v_2,v_3,v_4)} \psi(t,x,v_1)\,\alpha(v_{12341}) \times g^n(t,x,v_4)\,h^n(t,x,v_5)\,dv_{1234}\,dt\,dx$$

and in

$$\int_{(0,T)\times\Lambda}\int_{\mathbb{R}^3_{v_1}\times Z(v_2,v_3,v_4)} \psi(t,x,v_1)\,\alpha(v_{12341})\,f^n(t,x,v_1) \times g^n(t,x,v_2)\,g^n(t,x,v_3)\,dv_{1234}\,dt\,dx$$

By (A2) and Lemma 3.2,

$$\int (g^n-g)(t,x,v_4)\int_{\mathbb{R}^3_{v_1}\times Z(v_{234})} \alpha(v_{12341})\,h(t,x,v_1)\,dv_{123}\,dv_4$$

and

$$\int (h^n-h)(t,x,v_4)\left[\int_{Z(v_{234})} \alpha(v_{12341})\,dv_{23}\right]dv_1$$

strongly converge to zero in $L^1((0,T)\times\Lambda)$. By Egoroff's theorem, let $\sigma\subset(0,T)\times\Lambda$ be such that $|\sigma^c|$ is arbitrarily small and

$$\int (g^n-g)(t,x,v_4)\int_{\mathbb{R}^3_{v_1}\times Z(v_{234})} \alpha(v_{12341})\,h(t,x,v_1)\,dv_{123}\,dv_4$$

and

$$\int (h'' - h)(t, x, v_4) \left[\int_{Z(r_{234})} \alpha(v_{12341})\, dv_{23} \right] dv_1$$

converge to zero when n tends to infinity, uniformly with respect to $(t, x) \in \sigma$. Then

$$\left| \int_\sigma \int_{\mathbb{R}^3_{r_1} \times Z(r_{234})} \alpha(v_{12341})(g''(t, x, v_4)\, h''(t, x, v_1) - g(t, x, v_4)\, h(t, x, v_1))\, dv_{1234}\, dt\, dx \right|$$

$$\leqslant \int_\sigma \left| \int (g'' - g)(t, x, v_4) \int_{\mathbb{R}^3_{r_1} \times Z(r_{234})} \alpha(v_{12341})\, h(t, x, v_1)\, dv_{123}\, dv_4 \right| dt\, dx$$

$$+ \int_\sigma \int g''(t, x, v_4) \left| \int_{\mathbb{R}^3_{r_1}} (h'' - h)(t, x, v_1) \times \left[\int_{Z(r_{234})} \alpha(v_{12341})\, dv_{123} \right] dv_1 \right| dv_4\, dt\, dx$$

tends to zero when n tends to infinity. Indeed, (g'') is uniformly bounded in $L^1((0, T) \times \Lambda \times \mathbb{R}^3)$. Moreover, the convergence to zero of

$$\int_{\sigma^c} \int_{\mathbb{R}^3_{r_1} \times Z(r_{234})} \alpha(v_{12341})(g''(t, x, v_4)\, h''(t, x, v_1) - g(t, x, v_4)\, h(t, x, v_1))\, dv_{1234}\, dt\, dx$$

follows from the $L^1((0, T) \times \Lambda)$ weak compactness of

$$\int h'' \left(t, x, v_1 \int g''(t, x, v_4) \right) \left[\int_{Z(r_{234})} \alpha(v_{12341})\, dv_{23} \right] dv_4\, dv_1$$

Let us prove that

$$\lim_{n \to \infty} \int_{(0, T) \times \Lambda} \int_{\mathbb{R}^3_{r_1} \times Z(r_{234})} \alpha(v_{12341})[f''(t, x, v_1)\, g''(t, x, v_2)\, g''(t, x, v_3) - f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)]\, dv_{1234}\, dt\, dx = 0$$

By (A3), Lemma 3.2, and Egoroff's theorem, there is a set $\sigma \subset (0, T) \times \Lambda$ such that $|\sigma^c|$ is arbitrarily small and

$$\int_{v_3^2 \geq \Lambda - v_2^2} (g^n - g)(t, x, v_3) \left[\int_{v_4/v_4^2 = v_2^2 + v_3^2 - \Lambda} \alpha(v_{12341})\, dv_4 \right] dv_3$$

and

$$\int (f^n - f)(t, x, v_1) \left[\int_{v_4/v_4^2 = v_2^2 + v_3^2 - \Lambda} \alpha(v_{12341})\, dv_4 \right] dv_1$$

converge to zero when n tends to infinity, uniformly with respect to $(t, x) \in \sigma$. Restricting σ if necessary, we can also assume that $\int g(t, x, v)\, dv$ is bounded on σ. Then,

$$\left| \int_\sigma \int_{\mathbb{R}^3_{v_1} \times Z(v_{234})} \alpha(v_{12341}) [f^n(t, x, v_1)\, g^n(t, x, v_2)\, g^n(t, x, v_3) \right.$$
$$\left. - f(t, x, v_1)\, g(t, x, v_2)\, g(t, x, v_3)]\, dv_{1234}\, dt\, dx \right|$$

$$\leq \int_\sigma \int f^n(t, x, v_1)\, g^n(t, x, v_2) \left| \int_{v_3^2 \geq \Lambda - v_2^2} (g^n - g)(t, x, v_3) \right.$$
$$\left. \times \left[\int_{v_4/v_4^2 = v_2^2 + v_3^2 - \Lambda} \alpha(v_{12341})\, dv_4 \right] dv_3 \right| dv_2\, dv_1\, dt\, dx$$
$$+ \int_\sigma \int f^n(t, x, v_1)\, g(t, x, v_3) \left| \int_{v_2^2 \geq \Lambda - v_3^2} (g^n - g)(t, x, v_2) \right.$$
$$\left. \times \left[\int_{v_4/v_4^2 = v_2^2 + v_3^2 - \Lambda} \alpha(v_{12341})\, dv_4 \right] dv_2 \right| dv_3\, dv_1\, dt\, dx$$
$$+ \int_\sigma \int g(t, x, v_2) \int_{v_3^2 \geq \Lambda - v_2^2} g(t, x, v_3)$$
$$\times \left| \int_{\mathbb{R}^3_{v_1}} (f^n - f)(t, x, v_1) \right.$$
$$\left. \times \left[\int_{v_4/v_4^2 = v_2^2 + v_3^2 - \Lambda} \alpha(v_{12341})\, dv_4 \right] dv_1 \right| dv_3\, dv_2\, dt\, dx$$

tends to zero when n tends to infinity. Indeed,

$$\int_{\sigma}\int f^n(t,x,v_1)\,g^n(t,x,v_2)\,dv_{12}\,dt\,dx$$

$$\int_{\sigma}\int f^n(t,x,v_1)\,g(t,x,v_3)\,dv_{13}\,dt\,dx$$

and

$$\int_{\sigma}\left[\int g(t,x,v)\,dv\right]^2 dt\,dx$$

are bounded. Moreover, the convergence to zero of

$$\int_{\sigma'}\int_{\mathbb{R}^3_{r_1}\times Z(r_{234})}\alpha(v_{12341})[f^n(t,x,v_1)\,g^n(t,x,v_2)\,g^n(t,x,v_3)$$
$$-f(t,x,v_1)\,g(t,x,v_2)\,g(t,x,v_3)]\,dv_{1234}\,dt\,dx$$

follows from the $L^1((0,T)\times\Lambda)$ weak compactness of

$$\int_{\mathbb{R}^3_{r_1}\times Z(r_{234})}\alpha(v_{12341})\,f^n(t,x,v_1)\,g^n(t,x,v_2)\,g^n(t,x,v_3)\,dv_{1234}\,dt\,dx$$

Finally, it follows from (2.4), (1.8), (1.9), (1.12), and (1.13) that f and h belong to $L^{\infty}((0,T)\times\Lambda;L^1(\mathbb{R}^3_r))$ and (4.1) is satisfied.

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